PAPER

Fast Algorithm for Solving Matrix Equation in MoM Analysis of Large-Scale Array Antennas

Qiang CHEN†, *Regular Member*, **Qiaowei YUAN**††, *Nonmember*, *and* **Kunio SAWAYA**†, *Regular Member*

SUMMARY A new iterative algorithm based on the Gauss-Seidel iteration method is proposed to solve the matrix equation in the MoM analysis of the array antennas. In the new algorithm, the impedance matrix is decomposed into a number of sub matrices, which describe the self and mutual impedance between the groups of the array, and each sub matrix is regarded as a basic iteration unit rather than the matrix element in the ordinary Gauss-Seidel iteration method. It is found that the convergence condition of the ordinary Gauss-Seidel iteration scheme is very strict for the practical use, while the convergence characteristics of the present algorithm are greatly improved. The new algorithm can be applied to the sub domain MoM with a fast convergence if the grouping technique is properly used. The computation time for solving the matrix equation is reduced to be almost proportional to the square of the number of the array elements. The present method is effective in MoM analysis of solving large-scale array antennas.

key words: *antenna, antenna array, moment method, matrix equation, iterative method*

1. Introduction

The study of a large-scale phased array antenna is important in the rapidly developed mobile communication systems to provide broadband communication with high quality [1]. It is desired to analyze the basic array characteristics numerically, such as the active impedance and active element pattern, for designing the array antenna. The method of moment (MoM) is one of the efficient methods for the analysis of the array antennas. When the array antenna has N antenna elements and each element is divided into M segments for sub domain MoM analysis, $N_T \times N_T$ matrix equation must be solved to obtain the unknown current vector, where $N_T = M \times N$.

When the direct method such as the Gauss-Jordan method is employed to solve the matrix equation, the CPU time is proportional to N_T^3. In the case of a large-scale array antenna, N becomes so large that the CPU time for solving the matrix equation is much longer than that for evaluating the impedance matrix, which is proportional to N_T^2, and becomes the dominant part of the total CPU time in the MoM analysis. The direct method has another problem of the round-off error, which is relatively large, and can not be ignored for a large system of equations [2].

Instead of direct methods, iterative methods such as the Gauss-Seidel method and the Conjugate Gradient (CG) method have been applied to solve the linear matrix equations. The number of arithmetic operations of these iterative methods is usually proportional to N_T^2 for each iteration step. However, it was pointed out that the required number of the iteration step of the CG method depends on the analysis model and the size of segments of the basis functions, and is usually proportional to N_T, which means the total number of arithmetic operation is proportional to N_T^3, the same order as the direct method [3], [4], if the matrix-vector multiplication in the iteration of the CG method requires N_T^2 operations. Some methods to accelerate the matrix-vector multiplication have been proposed such as the fast multipole method (FMM) [5], multilevel fast multipole algorithm (MLFMA) [6], [7], and the fast inhomogeneous plane wave algorithm [8]. As for the Gauss-Seidel method, the criterion for the convergence of the iteration is that the diagonal values of the impedance matrix are large enough compared to the off-diagonal values [9], [10]. However, in most cases, the impedance matrix of the MoM analysis does not meet the requirements and the Gauss-Seidel iterative method does not converge. Therefore, it is necessary to develop a fast and stable iterative algorithm whose computation cost is less than the order of N_T^3 for solving the matrix equation in order to perform the MoM analysis of a large-scale array antenna.

In this paper, a new iterative algorithm based on the Gauss-Seidel method is proposed to solve the matrix equation $[Z][I] = [V]$ in the MoM analysis of the array antennas, whose CPU time is approximately proportional to N_T^2. The convergence criterion of the iterative algorithm is investigated and the effectiveness of the method is shown by some numerical examples.

2. Gauss-Seidel Scheme

The important procedure for solving the matrix equation $[Z][I] = [V]$ for unknown $[I]$ by using the Gauss-Seidel scheme is to split the matrix $[Z]$ into $[S]$ and $[T]$

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Fig. 1 Analysis model: A linear dipole array antenna.

so that the matrix equation becomes

$$[S][I] = -[T][I] + [V], \tag{1}$$

where $[S]$ contains the lower-left triangular part including the diagonal elements of $[Z]$, and $[T]$ contains the upper-right triangular part excluding the diagonal elements. The iterative scheme for solving Eq. (1) is given by:

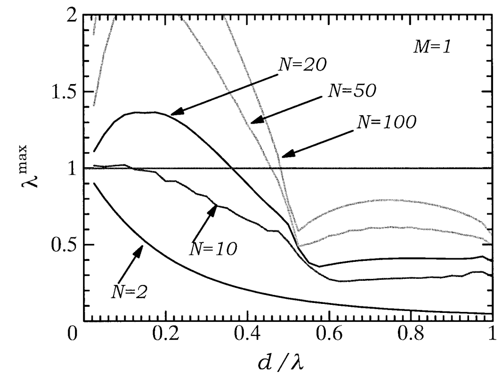
$$I_i^{(l_s+1)} = \frac{1}{S_{ii}}\left[V_i - \left(\sum_{j=1}^{i-1} S_{ij} I_j^{(l_s+1)} + \sum_{j=i+1}^{N} T_{ij} I_j^{(l_s)}\right)\right], \quad i = 1 \sim N;\ l_s = 1 \sim L_s. \tag{2}$$

where I_i, S_{ij} and T_{ij} are the elements of the vector $[I]$, matrices $[S]$ and $[T]$, respectively. The superscript l_s is the step number of the iteration. The initial $I_i^{(0)}$ is usually assumed to be zero. This iteration continues until $|I_i^{(l_s+1)} - I_i^{(l_s)}| \leq \epsilon$ for all i at the final L_sth step. The convergence criterion for the Gauss-Seidel scheme is that all the eigenvalues of the matrix $[S]^{-1}[T]$ have magnitudes less than unity [9].

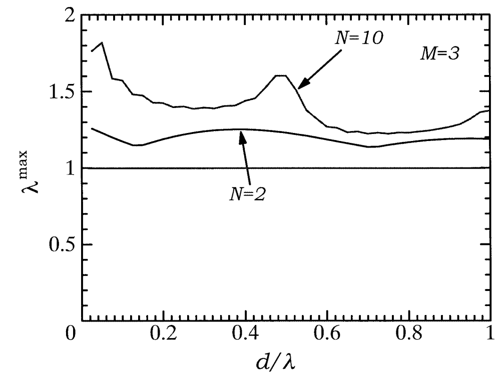
The piece-wise sinusoidal (PWS) MoM analysis [11] is performed to show whether the iteration method can be applied or not. The analysis model of the linear dipole antenna array is shown in Fig. 1. Each dipole element is divided into M overlapped dipole segments.

Figure 2 shows the largest magnitude of the eigenvalues of $[S]^{-1}[T]$ obtained from the PWS MoM analysis for a half wavelength dipole array when M is 1 and 3. Figure 2(a) shows the case of M=1, which means each dipole element is not divided into segments. The largest magnitude λ^{max} of the eigenvalues is smaller than unity for two cases, one is that d is larger than $\lambda/2$, and the other case is that d is smaller than $\lambda/2$ but N is limited to relatively small number, where λ is the wavelength. In both cases, the array elements are divided into the dipole segments. It is also indicated that the eigenvalues become large as the total number of the array elements N increases. Figure 2(b) shows that λ^{max} is always larger than unity when M is equal to 3 no matter what N is. It has been found by numerical analysis that λ^{max} is always lager than unity when M is larger than 1 if the Gauss-Seidel scheme is applied.

Figure 3 shows the largest magnitude of eigenvalues of matrix $[S]^{-1}[T]$ versus the array spacing d, when



(a) $M = 1$



(b) $M = 3$

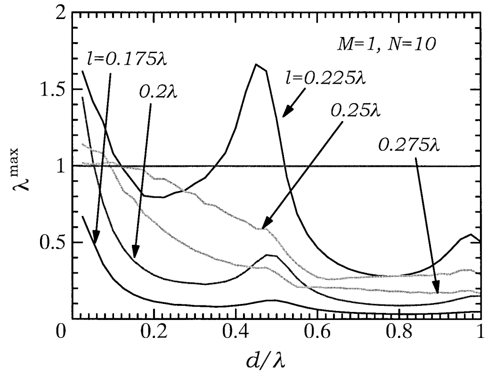
Fig. 2 The largest magnitude of eigenvalues of matrix $[S]^{-1}[T]$ versus array spacing d for the PWS MoM analysis of half wavelength dipole array.

the length of dipole element $2l$ changes. This figure indicates that the magnitude of the eigenvalues also depends on the length of the dipole antenna. For example, the eigenvalues are smaller than unity except for $l = 0.225\,\lambda$ in the case of M=1 and N=10, while the λ^{max} is larger than unity when the dipole length is larger than $0.225\,\lambda$, M=3 and N=10.

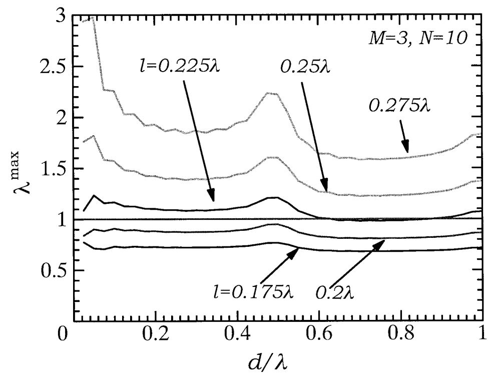
The above numerical results show that the Gauss-Seidel method can not be applied to solve the matrix equation in MoM for the usual array antennas. The convergence criterion for convergence depends on the total number of the array elements, the number of the segments for each element, the array spacing, and the geometry of the antenna. Therefore, it is necessary to improve the convergence characteristics of the iteration.

3. Novel Iterative Algorithm

In order to overcome the difficulty mentioned above, a novel iterative algorithm is proposed. In the novel iterative algorithm, the antenna array is divided into a number of groups and each group consists of several neighboring array elements, so that the impedance matrix can be decomposed into a number of sub matrices corresponding to the group of the array elements. The



(a) $M = 1$, $N = 10$



(b) $M = 3$, $N = 10$

Fig. 3 The largest magnitude of eigenvalues of matrix $[S]^{-1}[T]$ versus the array spacing d for the PWS MoM analysis of dipole array with various dipole lengths.

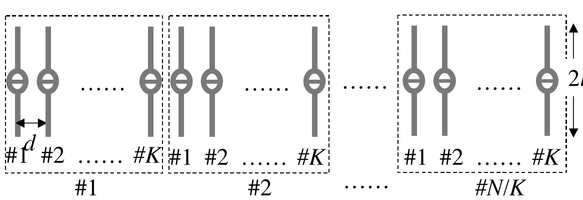


Fig. 4 Analysis model for the novel iterative algorithm: The linear dipole array antenna is divided into N/K groups.

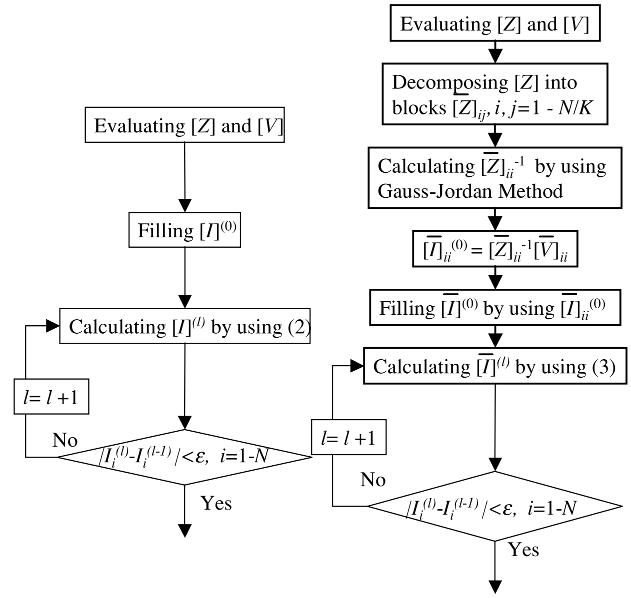


Fig. 5 Flowchart of MoM. (a) Matrix equation is solved using the Gauss-Seidel Method. (b) Matrix equation is solved using the proposed method.

diagonal sub matrices in the impedance matrix describe the self and mutual impedance between the divided unknown segments in the same group and the off-diagonal sub matrices include the mutual impedance between two divided unknown segments of different groups. The iterative unit is then changed to the sub matrices, and the sub matrices are the basic iteration units rather than the matrix element in the ordinary Gauss-Seidel iteration method. If each group consists of K elements, and the total array elements are divided into N/K groups completely as shown in Fig. 4, the iterating procedure is expressed by:

$$[\bar{I}]_i^{(l_s+1)} = [\bar{I}]_i^{(0)} - [\bar{Z}]_{ii}^{-1}\left[\sum_{j=1}^{i-1}[\bar{Z}]_{ij}[\bar{I}]_j^{(l_s+1)} + \sum_{j=i+1}^{N/K}[\bar{Z}]_{ij}[\bar{I}]_j^{(l_s)}\right]^T,$$
$$i = 1 \sim N/K;\ l_s = 1 \sim L_s. \tag{3}$$

where $[\bar{I}]_i$ is a MK current vector of the group i, and $[\bar{Z}]_{ij}$ is a $MK \times MK$ matrix, which means the self and mutual impedance of the dipole segments between two groups i and j and the inverse matrix $[\bar{Z}]_{ii}^{-1}$ is evaluated by using a direct method such as the Gauss-Jordan method. Notation $[\cdot]^T$ indicates the transposition of the matrix. For fast convergence, the initial $[\bar{I}]_i^{(0)}$ is assumed to be the current on a single group of the array elements ignoring the mutual coupling between the groups, which is given by

$$[\bar{I}]_i^{(0)} = [\bar{Z}]_{ii}^{-1}[\bar{V}]_i, \quad i = 1 \sim N/K. \tag{4}$$

where $[\bar{V}]_i$ is the voltage vector of group i.

The flowchart of the MoM using the normal Gauss-Seidel method and using the proposed method is given in Fig. 5 for comparison. It is clearly shown that the impedance matrix is decomposed into a number of sub matrices and the sub matrices become the iterative unit for the iteration process in the proposed method, which is different from the normal Gauss-Seidel methods.

The convergence criterion for this algorithm is investigated numerically by analyzing the same model shown in Fig. 4. First, the convergence characteristics are investigated in the case of $K = 1$, which means the array is not divided into groups.

Figure 6 shows the convergence criterion of the array spacing d versus the number of array elements N when the dipole length is $\lambda/2$. Although Fig. 6 shows

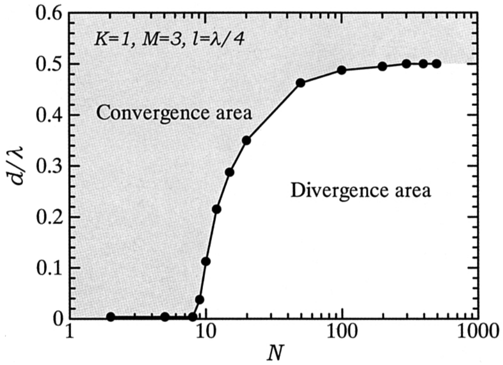


Fig. 6 Convergence criterion of array spacing d for various array element number N, when the dipole length is $l/2$.

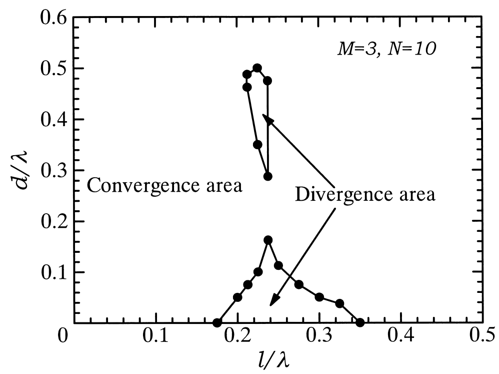


Fig. 7 Convergence criterion of minimum array spacing d for various lengths of dipole elements when M=3 and N=10.

the case of $K = 1$ and $M = 3$, it is found the convergence characteristics depend on the number of the array elements N and the array spacing d, but are almost independent of the segment number M of each array element. The figure illustrates that the novel iteration algorithm converges when d is larger than $\lambda/2$, or when d is smaller than $\lambda/2$ but N is limited to a relatively small number, which is similar to the case of $M = 1$ in the Gauss-Seidel method even though the array element is divided into several segments in the present analysis.

Although the increase of M may change the interaction between the divided segments in the same group greatly, the interaction between the segments of the different groups is not greatly influenced by the number M. Since the inverse of the sub matrices which describe the self and mutual impedance in the same group is evaluated by using the direct method such as the Gauss-Jordan method, the increase of M has little effect on the convergence of the iteration. That is why the convergence characteristics depend on the number of the array elements N and the array spacing d, but are almost independent of the segment number M of each array element, which is shown in Fig. 6.

Figure 7 shows the convergence criterion of array spacing d versus the length of the dipole element when

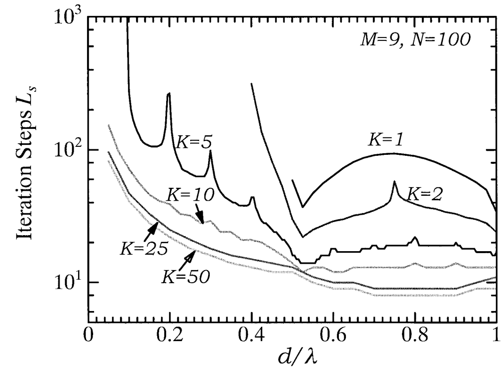


Fig. 8 Iteration steps required to perform Eq. (3) when every K elements are grouped, M=9 and N=100.

M=3 and N=10. The present method converges except when the length l is around $0.25\,\lambda$ and d is less than $0.5\,\lambda$, which is similar to the case of M=1 in the Gauss-Seidel method.

Although the convergence criterion of the present method is improved compared with that of the original Gauss-Seidel method, the divergence area still remains for K=1. Therefore, the convergence characteristics are examined for the case of $K > 1$.

Figure 8 shows the iteration steps required for Eq. (3) when M=9 and N=100. The curve of K=1, which means the grouping technique is not applied, is shown only in the range of 0.5 to 1 because the iteration diverges when d/λ is smaller than 0.5 as shown in Fig. 6. However, it is found that if the value of K increases to over 10, the iteration converges even when d/λ is as small as 0.04. Therefore, the grouping technique makes the iteration much more stable so that the iterative criterion is improved. When K increases, the curve of the required iteration steps becomes more and more flat, which means the required number of the iteration steps becomes independent of the array spacing. The singularity at d/λ=0.75 in the curve of K=2 is due to the strong coupling between the groups which appears periodically and decreases when d/λ becomes large.

Figure 9 shows the CPU time required to evaluate Eq. (3) when every K elements are grouped, and M=9, N=100. The value of the CPU time shown was measured by using a Pentium-III 450 MHz PC with 256 MB of memory. Although a large K can reduce the iteration steps and decrease the CPU time, a too large K would result in consuming a longer CPU time on the contrary. The total computational time T can be estimated by the expression

$$T = \alpha(KM)^3 + \beta L(MN/K)^2, \tag{5}$$

where the first term is for evaluating $[\bar{Z}]_{ii}^{-1}$, the second term is for iterating process, and the α, β are constants depending on the computer performance. Therefore, a large value of K can improve the convergence character-

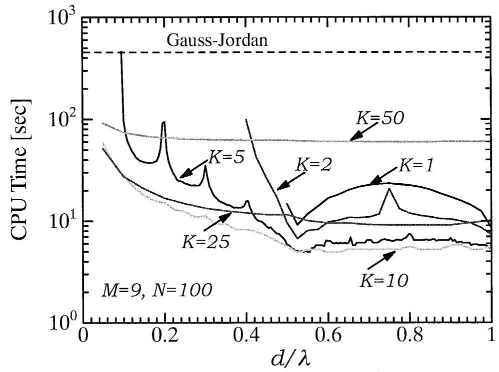


Fig. 9 CPU time required for Eq. (3) when every K elements are grouped, M=9 and N=100.

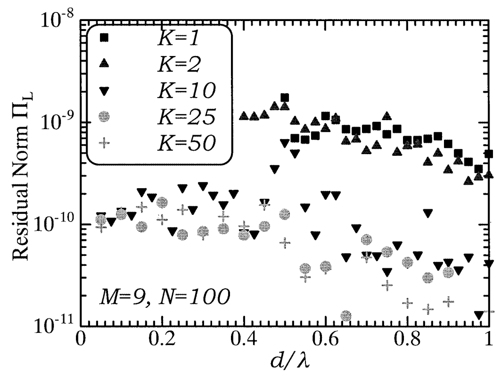


Fig. 10 Residual norm in the final iteration step versus array spacing.

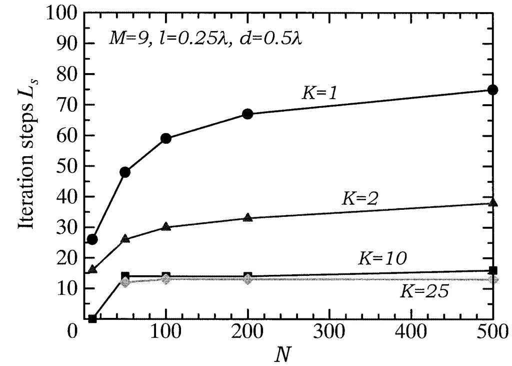


Fig. 11 Iteration steps versus total number of array elements with various K.

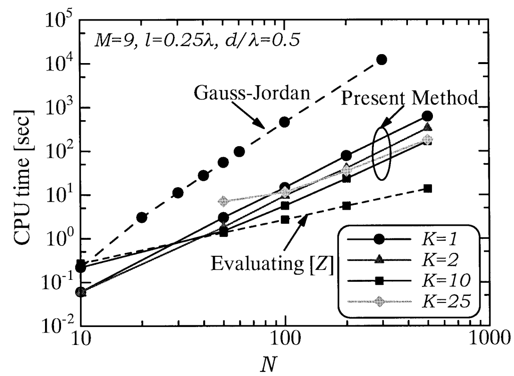


Fig. 12 CPU time versus total number of array elements with various K.

istics and decrease the required number of the iteration steps, but increase the CPU time for evaluating $[\bar{Z}]_{ii}^{-1}$ which is proportion to K to the third power. The CPU time used for the Gauss-Jordan method is also plotted in dashed line for comparison in Fig. 9. It can be concluded from the figure that a proper K value can greatly reduce the CPU time, while keeping the iteration process stable.

The error of I_L at the final iteration step L_s is estimated by the residual norm, which is defined by

$$\Pi_L = \frac{||ZI_L - V||}{||V||}. \tag{6}$$

This quantity is evaluated for various K and shown in Fig. 10, where the value of L_s is the same as shown in Fig. 8. Although a large K can lead to a small residual error, the residual norm is smaller than 2×10^{-9} for all cases, which indicates that good accuracy of the iteration is obtained.

The number of iteration steps versus the total number of the dipole array elements is shown in Fig. 11 for the case of $2l = \lambda/2$ and $d = \lambda/2$. When K is large, the number of iteration steps becomes independent of the element number N. If the number of iteration steps is independent of the element number N, the computational cost consumed by the present method would be approximately proportional to N^2 when N is as large as the value of second term shown in Eq. (5) is much greater than the value of the first term.

In order to show the validity of this method, the CPU time for solving the matrix equation versus N is shown in Fig. 12. The curve of the Gauss-Jordan method, one of the traditional iteration methods, is also plotted in the same figure for comparison. As expected, the CPU time is proportional to N^3 by using the Gauss-Jordan method, while it is proportional to N^2 by using the present method with a proper K. The cost saving effect of the numerical computation is significant. Also, the CPU time for evaluating the impedance matrix $[Z]$ is plotted in Fig. 12, which is smaller than that for solving the matrix equation when N is relatively large.

Figure 13 shows the input impedance of the central element of the 100-element dipole linear array when the array spacing varies. The results have been compared with the data obtained by using the normal Gauss-Jordan method. Since they agree completely, the results of the Gauss-Jordan method are not shown in the figure. The figure shows the vardility of the novel algorithm.

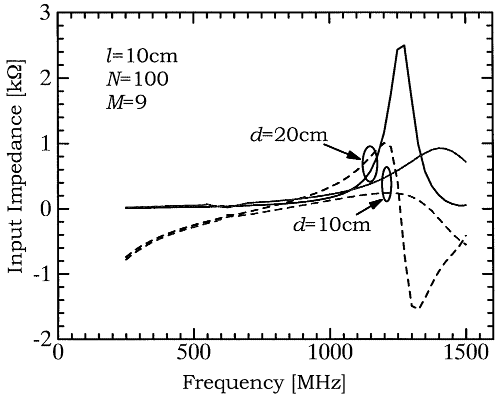


Fig. 13 Input impedance of the central element of 100-element dipole linear array.

4. Conclusion

An iterative algorithm has been proposed to solve the matrix equation of the MoM analysis for the array antenna. The convergence criterion of the iterative algorithm has been investigated numerically. The CPU time has been shown to be approximately proportional to N^2 when N is large enough, which has been greatly reduced compared with a direct method such as the Gauss-Jordan method. However, it is necessary to gather the array elements into groups properly to apply the method efficiently. How to group the array elements properly and decompose the impedance matrix mainly depends on the geometry of the array elements, array spacing, scan angle and so on, and is somewhat difficult to be determined. How to apply the grouping technique efficiently will be studied in the future.

It should be noted that the Gauss-Seidel method, which is applied to the present method, is not the most ideal method for solving the linear matrix equation. Some improving techniques such as the Back and Forth Seidel Process [12], the method of Successive Overrelaxation (SOR) [13] are superior to the original Gauss-Seidel method on the aspects of stability and convergence. However these techniques can also be directly applied to the present method.

This method is different from those fast algorithms such as the FMM [5] and MLFMA [6], [7] in two aspects. One is that this method is based on the Gauss-Seidel method, a linear iterative method, rather than the nonlinear method such as the CG method. The other is that the objective of this method is to decrease the iterative steps to save CPU time, while the FMM and MLFMA reduce the computational cost by decreasing complexity of the matrix-vector multiply appearing at each iteration step in the CG method.

Although linear arrays have been discussed in this paper, the method can be applied directly to planar arrays and 3-D problems if the impedance matrix is decomposed into a number of sub matrices properly according to the spatial distribution of the array elements.

The novel iterative algorithm is expected to enable the MoM analysis to be applied to large-scale array antennas.

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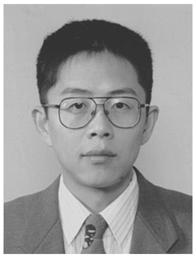

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