Fast Computation of Layered Media Green's Function via Recursive Taylor Expansion

Keisuke Konno, *Member, IEEE*, Qiang Chen, *Senior Member, IEEE*, and Robert J. Burkholder, *Fellow, IEEE*

***Abstract*—The layered media Green's function (LMGF) is useful as a kernel of the method of moments (MoM) for a thin stratified medium. The self/mutual impedance expression of the MoM in conjunction with the LMGF includes a semi-infinite spectral integral inside the multiple integrals over spatial variables. As a result, computation of impedance matrix entries is quite costly. In this letter, an interpolation method by using the Taylor expansion is proposed. The method precomputes the spectral integral for a single spatial variable. The derivatives in the expansion are found in closed form using the recursive property of the Bessel functions. After that, the precomputed results of the spectral integral are interpolated via the Taylor expansion when the multiple integrals over spatial variables are performed. Due to the proposed method, the spectral integral and multiple integrals over spatial variables are separated from each other, and the resultant CPU time becomes quite manageable for very large problems.**

***Index Terms*—Bessel function, interpolation, layered media Green's function (LMGF), method of moments (MoM), Taylor expansion.**

I. Introduction

Method of moments (MoM) is a quite powerful technique for numerical analysis of antennas and scatterers [1]. In order to deal with a microstrip antenna by using the MoM, a layered media Green's function (LMGF) is used [2]–[4]. The LMGF includes the so-called Sommerfeld integral (SI), i.e., a semi-infinite spectral integral, and the numerical evaluation of the SI is quite costly. In addition to the evaluation of the SI, multiple integrals over spatial variables must be evaluated when self/mutual impedance between source and observation segments is obtained. The SI must be recomputed every time when multiple integrals over spatial variables are evaluated because the integrand of the SI depends on both spectral and spatial variables. As a result, numerical evaluation of the self/mutual impedance by using the LMGF is inefficient in a direct manner.

In order to evaluate the self/mutual impedance efficiently, various techniques have been proposed. A discrete complex image method (DCIM) is one of the popular techniques to evaluate the SI efficiently [5]–[7]. The SI is evaluated in the spatial domain by using complex exponentials via the Sommerfeld identity when the DCIM is used. As a result, the SI is evaluated efficiently because spatial domain counterparts of the SI converge quickly. However, it is known that the far-field prediction of the DCIM is poor. Moreover, it is difficult to apply the DCIM to the SI when its spatial domain counterparts are unknown. On the other hand, interpolation methods are another approach to evaluate the self/mutual impedance efficiently. Polynomial interpolation techniques such as a spline technique are well known [8]–[10]. The unknown coefficients of the polynomials are numerically obtained as a solution of a linear system. However, very few papers that deal with interpolation techniques for generalized multilayered problems have been published [11]. In addition, a high-order interpolation technique, which is based on a purely analytic scheme rather than a numerical solution, is hardly seen.

In this letter, a novel interpolation technique using the Taylor expansion is proposed for the efficient evaluation of the self/mutual impedance using the LMGF. Bessel functions and their derivatives multiplied by a propagation factor are obtained and tabulated for a single spatial variable before multiple integrals over the spatial variables are performed. The SI for a specific spatial variable is evaluated by using interpolation via the Taylor expansion of a Bessel function of an arbitrary order. The proposed interpolation technique is straightforward and easy to implement because the Taylor expansion is performed using derivatives of the Bessel functions that can be obtained recursively. Moreover, the proposed interpolation technique is applicable to general multilayer problems, and its efficiency improves as the size of problems increases.

II. Proposed Interpolation Method

As an example, a microstrip dipole array antenna embedded in a double-layered medium backed by a ground plane is shown in Fig. 1. For the multilayered medium, an LMGF is expressed using TE and TM waves propagating in the medium as follows:

$$G^{\mathrm{TE}}_{yy'}(\mathbf{r},\mathbf{r}') = \frac{j}{8\pi}\int_0^\infty \left\{\frac{1}{k_{mz}}\frac{2\cos 2\phi}{\rho}J_1(k_\rho \rho) - \frac{k_\rho}{k_{mz}}(1+\cos 2\phi)J_0(k_\rho\rho)\right\}F^{\mathrm{TE}}(k_\rho, z, z')dk_\rho \quad (1)$$

$$G^{\mathrm{TM}}_{yy'}(\mathbf{r},\mathbf{r}') = -\frac{j}{8\pi}\int_0^\infty \left\{\frac{1}{k_{mz}}\frac{2\cos 2\phi}{\rho}J_1(k_\rho \rho) + \frac{k_\rho}{k_{mz}}(1-\cos 2\phi)J_0(k_\rho\rho)\right\}\frac{\partial^2 F^{\mathrm{TM}}(k_\rho, z, z')}{\partial z \partial z'}dk_\rho \quad (2)$$

Manuscript received August 26, 2016; accepted October 18, 2016. Date of publication October 19, 2016; date of current version May 1, 2017. This work was supported in part by JSPS KAKENHI under Grants 25420394 and 26820137, and in part by JSPS Postdoctoral Fellowships for Research Abroad.

K. Konno and Q. Chen are with the Department of Electrical and Communications Engineering, Tohoku University, Sendai 8577, Japan (e-mail: konno@ecei.tohoku.ac.jp; chenq@ecei.tohoku.ac.jp).

K. Konno and R. J. Burkholder are with the Department of Electrical and Computer Engineering, ElectroScience Laboratory, The Ohio State University, Columbus, OH 43212 USA (e-mail: burkholder.1@osu.edu).

Color versions of one or more of the figures in this letter are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/LAWP.2016.2619483

1536-1225 © 2016 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission.
See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.

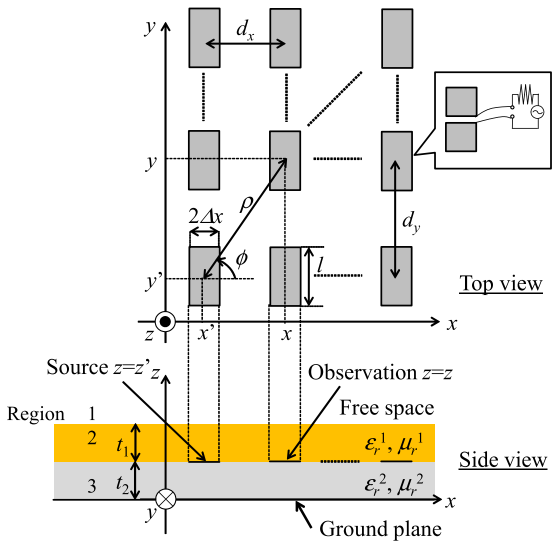

Fig. 1. Microstrip dipole array antenna embedded in a double-layered medium backed by a ground plane.

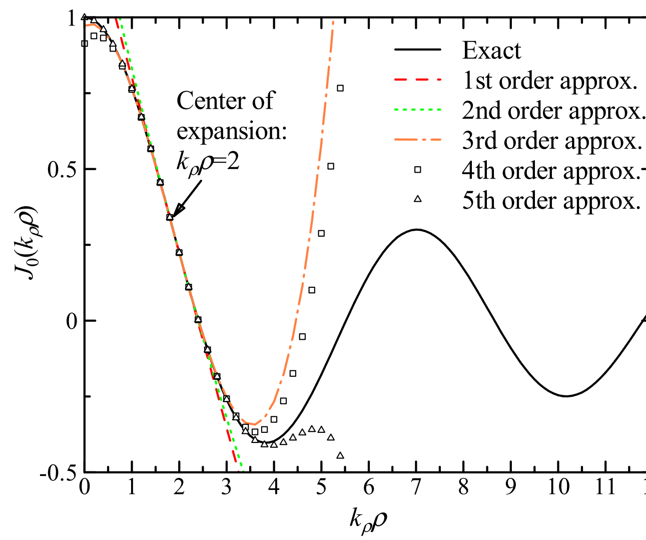

Fig. 2. Bessel function of the first kind of order 0 interpolated by using the Taylor expansion (center of expansion is $k_\rho \rho = 2$).

where $G_{yy'}^{\mathrm{TE}}$ and $G_{yy'}^{\mathrm{TM}}$ are the $y - y'$ components of the dyadic Green's function of the layered medium for TE and TM waves, respectively. J_0 and J_1 are the Bessel functions of the first kind of order 0 and 1, respectively. F^{TE} and F^{TM} are spectral propagation factors characterizing the layered media corresponding to TE and TM waves, respectively, and $\rho = \sqrt{(x-x')^2 + (y-y')^2}$, $\phi = \arctan(\frac{y-y'}{x-x'})$, and $k_{mz} = \sqrt{k_m^2 - k_\rho^2}$, where k_m is the wavenumber of mth region.

The self/mutual impedance between two segments is expressed by using the LMGF as

$$Z_{ij}^{\mathrm{TE/TM}} = j\omega\mu_0 \iint_S \mathbf{J}_y(x,y) \cdot \iint_{S'} \mathbf{J}_{y'}(x',y') \times G_{yy'}^{\mathrm{TE/TM}}(\mathbf{r},\mathbf{r}')dx'dy'dxdy. \quad (3)$$

As shown in (3), $Z_{ij}^{\mathrm{TE/TM}}$ is composed of fourfold integrals of the LMGF in spatial coordinates, requiring evaluation of the computationally expensive semi-infinite spectral integral of (1) or (2). The fourfold integrals over spatial variables and the semi-infinite spectral integral in the LMGF cannot be separable because the LMGF depends not only on the spectral variable, but also on the spatial variables. Therefore, the resultant CPU time for numerical integration is quite long.

As shown in (1) and (2), the spatial variables ρ and ϕ can be moved to the outside of the semi-infinite spectral integral except for ρ in the Bessel functions. Therefore, the number of times the semi-infinite spectral integral needs to be computed can be reduced if the Bessel functions are interpolated with respect to ρ. Here, the Bessel functions of the first kind of order n are expressed by using the well-known Taylor expansion

$$J_n(k_\rho\rho) = J_n(k_\rho\rho_i) + \frac{\partial J_n(k_\rho\rho_i)}{\partial\rho}(\rho-\rho_i) + \frac{1}{2}\frac{\partial^2 J_n(k_\rho\rho_i)}{\partial\rho^2}(\rho-\rho_i)^2 + \frac{1}{6}\frac{\partial^3 J_n(k_\rho\rho_i)}{\partial\rho^3}(\rho-\rho_i)^3 + \cdots. \quad (4)$$

Derivatives of the Bessel functions to arbitrary order can be obtained recursively [13]. Therefore, interpolation of the Bessel functions by using the Taylor expansion is straightforward and easy to implement. Of course, asymptotic expansions of the Bessel functions are often used when $k_\rho\rho \gg 1$ and can be interpolated in the same manner.

As a numerical example, the Bessel function of the first kind of order 0 is interpolated by using the Taylor expansion in Fig. 2 for small arguments. The interpolation using the Taylor expansion works well in the proximity of the center of expansion and improves with higher order terms. Moreover, it is observed that interpolation using the Taylor expansion works even when $k_\rho\rho$ is small. Therefore, the interpolation technique can be applied not only to the computation of mutual impedance, but also to the computation of the self-impedance.

In our proposed interpolation technique, (4) is substituted into the LMGFs of (1) and (2) and the individual terms are precomputed at discrete values of ρ. Then, the LMGFs can be quickly evaluated for arbitrary ρ. The algorithm is summarized as follows.

1) The semi-infinite spectral integral is truncated at $k_\rho^{\max}$ in order to obtain sufficiently convergent results.
2) $\rho_{\min}$ and $\rho_{\max}$ that are the minimum and maximum values of ρ inside the double surface integrals are calculated, respectively.
3) The number of sampling points L is obtained from $L = \frac{\rho_{\max} - \rho_{\min}}{\Delta\rho}$. Here, $\Delta\rho$ satisfies $k_\rho^{\max}\Delta\rho < a$, where a is

an arbitrary limit of interpolation of the Taylor expansion around one sampling point.

4) The semi-infinite spectral integrals over the Bessel functions and their derivatives multiplied by the propagation factors are precomputed and tabulated for $\rho_i (i = 1, 2, \ldots, L)$, where ρ_i is the ith sampling point.
5) The double surface integrals over the LMGF that is interpolated via the Taylor expansion are computed by using (3) to yield the self/mutual impedance.

The computational cost of the above-mentioned algorithm depends on the total number of elements in the impedance matrix N^2, the number of quadrature integration points for the spatial and spectral integrals L_s and L_k, respectively, the number of interpolation sampling points L, and the number of multiplications for computation of the higher order derivatives of the Bessel functions L_b, which is approximately proportional to their order of derivatives. Without interpolation, the matrix fill time is $O(L_s L_k N^2)$. With interpolation, the cost of the precomputation of the spectral integrals is $O(L_b L L_k N^2)$, and the cost of the subsequent matrix fill is $O(L_s N^2)$. The total cost is therefore $O((L_b L L_k + L_s)N^2) \approx O(L_s N^2)$ because the matrix fill time is still much larger than the precomputation time. We see that the computational cost of evaluating (3) with the proposed interpolation method is expected to be reduced by a factor of L_k compared to direct evaluation.

III. Numerical Simulation

In order to verify the performance of the proposed interpolation technique, the microstrip dipole array antenna, shown in Fig. 1, is numerically analyzed. The self/mutual impedance matrix entries are obtained by using the MoM with the LMGF. Piecewise sinusoidal basis functions are used as basis/testing functions for the printed metallic elements. The fourfold integrals over space coordinates x, x', y, and y' are transformed into a sum of double integrals via coordinate transformation [12]. The effect of a ground plane is included in the numerical simulation by using image theory. In order to improve the convergence of the semi-infinite spectral integral, the direct wave was extracted from the LMGF and its contribution was evaluated via Green's function for a homogeneous medium. As a result of a convergence study, the semi-infinite spectral integral was truncated at $k_\rho^{\max} = k_{\max} + 2$ where $k_{\max}$ is the maximum wavenumber in a medium. A Taylor expansion of order 5 was used for interpolation and a is set to 2. Asymptotic expansions of the Bessel functions are used when $k_\rho \rho > 4$.

Input impedance of a single dipole antenna embedded in a double-layered medium is shown in Fig. 3. As a reference, the input impedance obtained by using the commercial simulator FEKO is shown. The calculated error between the original and the interpolated input impedance is only 0.1% and verifies the excellent agreement between two curves quantitatively.

The CPU time for computation of the impedance matrix entries of array antennas is shown in Fig. 4. Fig. 4 shows that the MoM with the proposed method is 28–60 times faster than that without the proposed method. In particular, the speed-up ratio increases as the size of the array antenna increases. This is because the number of quadrature points L_k for the spectral integral increases as the spacing between source and observation points increases because its integrand is a periodic function of period $\frac{2\pi}{\rho}$ asymptotically. As a result, the efficiency of the proposed method is enhanced as the size of the problem increases.

In order to verify the performance of the proposed method clearly, it is compared to a two-point C1 (slope-continuous) polynomial interpolation method (see the Appendix). Impedance matrix entries and the total CPU time for matrix filling are shown in Table I. Here, interpolation is performed

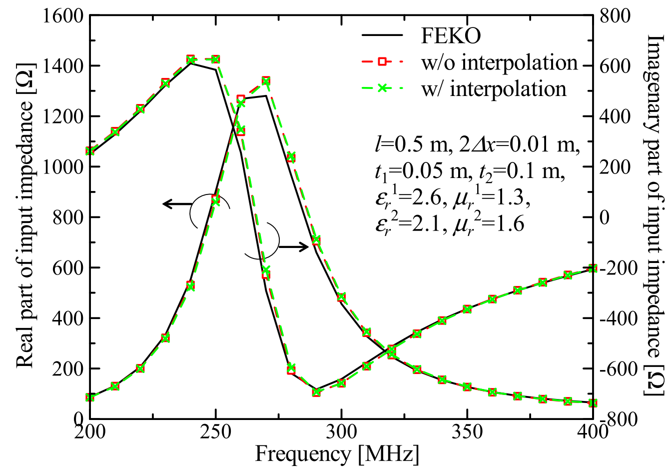

Fig. 3. Input impedance of a single microstrip dipole antenna embedded in a double-layered medium.

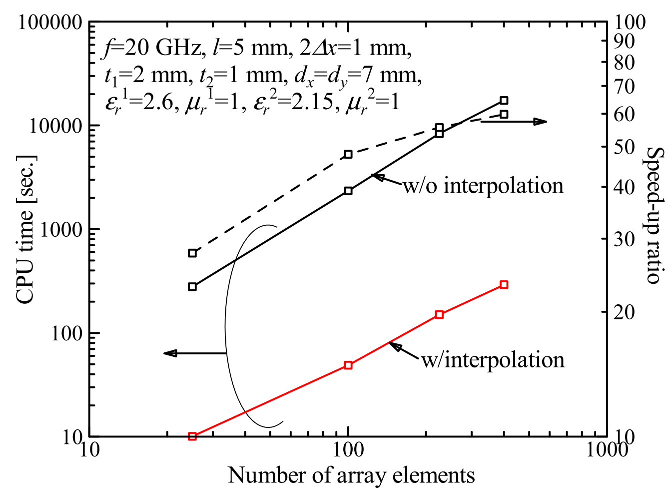

Fig. 4. CPU time for computation of impedance matrix entries.

TABLE I
Matrix Entries and Total CPU Time for Matrix Filling for a Single Dipole Antenna Divided into Seven Segments

Method	W/o interpolation	Proposed	Polynomial
$Z_{1,2}^{TE/TM}$	$-0.53 + j0.058$	$-0.55 + j0.057$	$-0.53 + j0.058$
$Z_{1,5}^{TE/TM}$	$-0.13 + j0.42$	$-0.13 + j0.42$	$-1.43 - j2.28$
$Z_{1,7}^{TE/TM}$	$0.2 + j0.55$	$0.2 + j0.56$	$-8.0 - j4.70$
CPU time	11.54 s	0.31 s	0.33 s

f = 200 MHz, l = 0.5 m, $2\Delta x$ = 0.01 m, t_1 = 0.05 m, t_2 = 0.1 m, ϵ_r^1 = 2.6, μ_r^1 = 1.3, ϵ_r^2 = 2.3, and μ_r^2 = 1.6.

with the minimum number of sampling points, i.e., one point for the proposed method and two points for the polynomial interpolation method. The other parameters are completely the same in order to impartially compare their performances. It is found that the accuracy of both the methods is different, whereas the CPU time of both the methods is the same. For example, relative errors of impedance matrix entries obtained by using the proposed interpolation method are less than 4%. On the other hand, relative errors of impedance matrix entries obtained by using the polynomial interpolation method can be quite large. Accuracy of the polynomial interpolation method is expected to be enhanced when the number of sampling points or the order of polynomials increase. However, the resultant CPU time also increases. In addition, the polynomial interpolation method may suffer from singularities as shown in (8)–(11) because the spacing between two sampling points becomes closer to each other as the number of sampling points increase. Therefore, here the proposed method is considered superior to the C1 two-point polynomial interpolation method.

IV. Conclusion

In this letter, a novel interpolation method for computation of the self/mutual impedance of microstrip antenna arrays with the LMGF has been proposed. The proposed method is based on a Taylor expansion of the Bessel functions exploiting the recursive computation of the derivatives. Therefore, the proposed method is simple, straightforward, and easy to implement for general multilayered problems. Numerical simulations demonstrated that the efficiency of the proposed method increases as the size of the problem increases. The proposed method is easily applicable to fast solvers such as the fast multipole method in which the impedance matrix entries for self and adjacent groups must be obtained explicitly.

Appendix
Polynomial Interpolation

Let us consider the following spectral integral:

$$S_0^{\mathrm{TE}}(\rho) = \int_0^{\infty} \frac{k_\rho}{k_z^m} J_0(k_\rho \rho) F^{\mathrm{TE}}(k_\rho, z, z') dk_\rho. \tag{5}$$

Here, it is assumed that $S_0^{\mathrm{TE}}(\rho_1), S_0^{\mathrm{TE}}(\rho_2), \frac{S_0^{\mathrm{TE}}(\rho_1)}{\partial \rho}$, and $\frac{S_0^{\mathrm{TE}}(\rho_2)}{\partial \rho}$ are known where ρ_1 and ρ_2 are two consecutive samples of the radial distance ρ. Between these two samples, the spectral integral can be interpolated using a third-order polynomial that enforces C1 (slope) continuity

$$S_0^{\mathrm{TE}}(\rho) = a_0 + a_1\rho + a_2\rho^2 + a_3\rho^3 \tag{6}$$

where a_0–a_3 are unknown coefficients. In order to obtain these unknown coefficients, the following 4×4 matrix equation is solved:

$$\begin{bmatrix} 1 & \rho_1 & \rho_1^2 & \rho_1^3 \\ 1 & \rho_2 & \rho_2^2 & \rho_2^3 \\ 0 & 1 & 2\rho_1 & 3\rho_1^2 \\ 0 & 1 & 2\rho_2 & 3\rho_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{7}$$

where $b_0 = S_0^{\mathrm{TE}}(\rho_1), b_1 = S_0^{\mathrm{TE}}(\rho_2), b_2 = \frac{S_0^{\mathrm{TE}}(\rho_1)}{\partial \rho}$, and $b_3 = \frac{S_0^{\mathrm{TE}}(\rho_2)}{\partial \rho}$. Finally, the solutions for the unknown coefficients are obtained analytically via algebraic operations

$$\begin{aligned} a_0 = &-\frac{b_2\rho_1(2\rho_1^2 + 4\rho_1\rho_2 + \rho_2^2) + b_3\rho_1^2(4\rho_1 + 3\rho_2)}{(\rho_1 - \rho_2)^2} \\ &- \frac{(b_0 - b_1)\rho_1^2(\rho_1 - 3\rho_2)}{(\rho_1 - \rho_2)^3} + b_0 \end{aligned} \tag{8}$$

$$\begin{aligned} a_1 = &\frac{b_2\rho_2(2\rho_1 + \rho_2) + b_3\rho_1(\rho_1 + 2\rho_2)}{(\rho_1 - \rho_2)^2} \\ &- \frac{6(b_0 - b_1)\rho_1\rho_2}{(\rho_1 - \rho_2)^3} \end{aligned} \tag{9}$$

$$\begin{aligned} a_2 = &-\frac{b_2(\rho_1 + 2\rho_2) + b_3(2\rho_1 + \rho_2)}{(\rho_1 - \rho_2)^2} \\ &+ \frac{3(b_0 - b_1)(\rho_1 + \rho_2)}{(\rho_1 - \rho_2)^3} \end{aligned} \tag{10}$$

$$a_3 = \frac{b_2 + b_3}{(\rho_1 - \rho_2)^2} - 2\frac{b_0 - b_1}{(\rho_1 - \rho_2)^3}. \tag{11}$$

Other kinds of spectral integrals can also be interpolated in the same manner.

Acknowledgment

The authors would like to thank staff in the Cyberscience Center, Tohoku University, for their helpful advice.

References

[1] R. F. Harrington, *Field Computation by Moment Methods*. New York, NY, USA: Macmillan, 1968.
[2] N. K. Das and D. M. Pozar, "A generalized spectral-domain Green's function for multilayer dielectric substrates with application to multilayer transmission lines," *IEEE Trans. Microw. Theory Techn.*, vol. MTT-35, no. 3, pp. 326–335, Mar. 1987.
[3] W. C. Chew, J. L. Xiong, and M. A. Saville, "A matrix-friendly formulation of layered medium Green's function," *IEEE Antennas Wireless Propag. Lett.*, vol. 5, pp. 490–494, 2006.
[4] Y. P. Chen, W. C. Chew, and L. Jiang, "A new Green's function formulation for modeling homogeneous objects in layered medium," *IEEE Trans. Antennas Propag.*, vol. 60, no. 10, pp. 4766–4776, Oct. 2012.
[5] D. G. Fang, J. J. Yang, and G. Y. Delisle, "Discrete image theory for horizontal electric dipoles in a multilayered medium," *Proc. IEE H, Microw., Antennas, Propag.*, vol. 135, no. 5, pp. 297–303, Oct. 1988.
[6] A. Alparslan, M. I. Aksun, and K. A. Michalski, "Closed-form Green's function in planar layered media for all ranges and materials," *IEEE Trans. Microw. Theory Techn.*, vol. 58, no. 3, pp. 602–613, Mar. 2010.
[7] Y. P. Chen, W. C. Chew, and L. Jiang, "A novel implementation of discrete complex image method for layered medium Green's function," *IEEE Antennas Wireless Propag. Lett.*, vol. 10, pp. 419–422, 2011.
[8] F. A.-Monferrer, A. A. Kishk, and A. W. Glisson, "Green's functions analysis of planar circuits in a two-layer grounded medium," *IEEE Trans. Antennas Propag.*, vol. 40, no. 6, pp. 690–696, Jun. 1992.
[9] G. Valerio, P. Baccarelli, S. Paulotto, F. Frezza, and A. Galli, "Efficient near-field interpolation of mixed-potential Green's functions in layered media," *IEEE Antennas Wireless Propag. Lett.*, vol. 8, pp. 674–677, 2009.
[10] G. Valerio, P. Baccarelli, S. Paulotto, F. Frezza, and A. Galli, "Regularization of mixed-potential layered-media Green's functions for efficient interpolation procedures in planar periodic structures," *IEEE Trans. Antennas Propag.*, vol. 57, no. 1, pp. 122–134, Jan. 2009.
[11] P. R. Atkins and W. C. Chew, "Fast computation of the dyadic Green's function for layered media via interpolation," *IEEE Antennas Wireless Propag. Lett.*, vol. 9, pp. 493–496, 2010.
[12] A. Köksal and J. F. Kauffman, "Mutual impedance of parallel and perpendicular coplanar surface monopoles," *IEEE Trans. Antennas Propag.*, vol. 39, no. 8, pp. 1251–1256, Aug. 1991.
[13] C. A. Balanis, *Advanced Engineering Electromagnetics*, 2nd ed. Hoboken, NJ, USA: Wiley, pp. 967–979.